Statistical and Low Temperature Physics (PHYS393)

6. Liquid Helium-4

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1. Fritz London's explanation of superfluidity in liquid helium-4 using Bose Einstein condensate (BEC).

2. Theory and experimental search for BEC.

3. Laszlo Tisza's 2-fluid model theory. Experiments.

4. Lev Landau's critical velocity experiments.

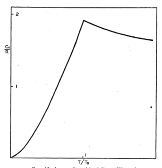
5. Explanation on perfect conductivity. Second sound.

Explaining superfluidity.

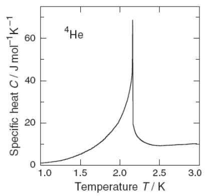
Fritz London suggested using Bose Einstein condensate (BEC) to explain superfluidity. This are possible for 2 reasons:

1. Helium-4 is a boson, so it may undergo Bose Einstein condensation.
2. The liquid would then become a wavefunction, so it would show no viscosity unless it flows with enough energy.

In order to test his theory, London calculated the transition temperature and heat capacity of the BEC (left figure). He showed that there is some agreement with liquid helium-4 (right figure), at least in trend.



Specific heat of an ideal Bose-Einstein gas.



Left: Enss and Hunklinger, Low Temperature Physics, page 8, 2005.
Right: London, Physical Review, volume 54, page 947, 1938

BEC heat capacity.

We shall look at a simplified version of London's treatment to calculate the BEC heat capacity.

This involves finding:

1. the temperature at which condensation starts,

2. the change in number of atoms with temperature, and

3. the heat capacity below condensation temperature.

When do the bosons condense?

Using the Bose-Einstein distribution that we have learnt, we can calculate the temperature at which bosons condense into BEC.

Recall the expression we first derived for the Bose-Einstein distribution:

$$\frac{n_i}{g_i} = \frac{1}{\exp(-\lambda_1 - \lambda_2\varepsilon_i) - 1}$$

When applying this to phonons, λ_1 was left out because the number of phonons is not fixed.

Now that we are dealing with atoms, which are real bosons, the number is fixed and we have to keep λ_1. This is usually written in terms of μ as follows:

$$n(\varepsilon)d\varepsilon = \frac{g(\varepsilon)d\varepsilon}{\exp((\varepsilon - \mu)/k_BT) - 1}$$

The density of states

We have seen a number of different density of states.

For electrons, we need to multiply by 2 because of the spin states. For phonons, we multiply by 3 for the three possible polarisations. For photons, we multiply by 2 for the two polarisation states.

So what do we do for atoms that are bosons?

The answer is: we use the same density of state for the ideal gas:

$$g(\varepsilon) = \frac{4m\pi V}{h^3}(2m\varepsilon)^{1/2}$$

This is the very first one that we have seen, before we have to include the additional effects of spin and polarisation.

The chemical potential

μ is called the chemical potential. It is determined by the total number of particles. We get this by integrating:

$$N = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp((\varepsilon - \mu)/k_BT) - 1}$$

I have assumed that the ground state energy is zero, which is why I integrated from zero.

This is not easy to solve. Instead, we shall make use of some approximations at low temperature.

For a start, note that μ could depend on temperature, since the above equation contains T.

The occupation number

We focus on the occupation number in the integral:

$$f(\varepsilon) = \frac{1}{\exp((\varepsilon - \mu)/k_B T) - 1}$$

First, note that μ cannot be positive, or else for small energy the exponential function would be less than one. Then the denominator would be negative. This means negative occupation number, which is not physical.

Next, we know that when temperature is low enough, a large number of atoms would go into the ground state $\epsilon = 0$. The number would be as large as N, which is of the order of 1 mole ($\approx 10^{24}$).

This means that $f(0)$ would be very large, so that μ must be very close to zero. Then the exponential function would be close to 1 and the denominator would be close to zero.

Number of atoms in the excited state

So at very low temperature, it is safe to assume that $\mu = 0$.
We can then write

$$N_{ex} = \int_0^\infty \frac{g(\varepsilon) d\varepsilon}{\exp(\varepsilon/k_B T) - 1}$$

I have added a subscript ex to N. The reason will become clear in a moment. First, integrate this with the help of the Table of Integrals:

$$N_{ex} = \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} 2.612 V$$

Notice that T is in the denominator. This suggests that the total number of atoms decreases with temperature - that they are disappearing !

In fact, the above integral includes only atoms above the ground state $\varepsilon = 0$. The "missing" atoms are in fact going into the ground state.

The condensation temperature

The formula for N_{ex} is only able to compute the number of particles above the ground state. This is because the density of states $g(\varepsilon)$ in the integral is zero when $\varepsilon = 0$.

When a substantial fraction of the particles start going into the ground state, this has to be added separately. So N_{ex} is the number of atoms in the excited states. Hence the subscript ex.

We know that as soon as N_{ex} becomes less than the original number N, then $(N - N_{ex})$ atoms start going into the ground state.

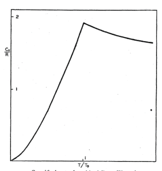
Therefore, condensation takes place when $N_{ex} = N$.
Substituting this and solving for T, we get

$$T_{BE} = \frac{h^2}{2\pi m k_B} \left(\frac{N}{2.612 V}\right)^{2/3}.$$

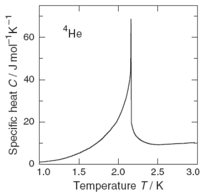
Transition temperature

Substituting the mass of helium atom and the molar volume into the formula for the condensation temperature T_{BE}, we find 3.13 K.

This is fairly close to the lambda point temperature of 2.18 K, at which liquid helium-4 changes to a superfluid,



Specific heat of an ideal Bose-Einstein gas.

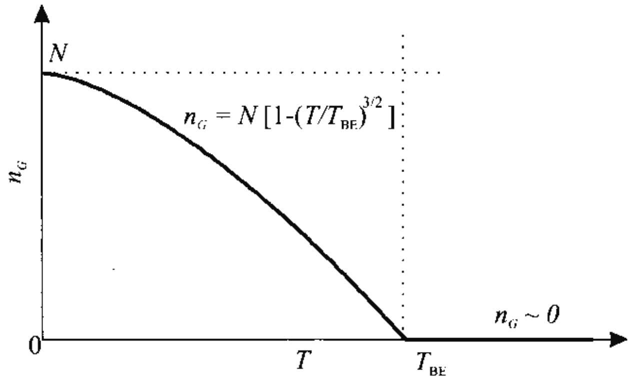


and provides some support for the idea that the superfluid is a BEC.

Number of atoms in the ground state

Above T_{BE}, there are very little atoms in the ground state.

Below T_{BE}, this number increases until all atoms fall into the ground state.



Glazer and Wark, Statistical mechanics, p. 101

Heat capacity.

Next, to calculate the heat capacity, recall the expression for the number of excited particles:

$$N_{ex} = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_BT) - 1}.$$

To find the heat capacity, we first find the energy U. The above integral is a sum over particle number in every energy interval.

To find U, we need to multiply by the energy at each interval:

$$U = \int_0^\infty \frac{\varepsilon g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_BT) - 1}.$$

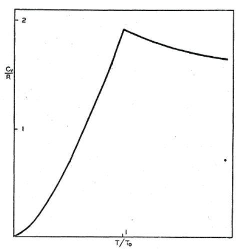
Integrating using the table of integrals gives

$$U = 0.7704 k_B N \frac{T^{5/2}}{T_{BE}^{3/2}}.$$

Below condensation temperature.

The heat capacity below the condensation temperature is then obtained by differentiating with respect to T:

$$C = 1.926 k_B N \left(\frac{T}{T_{BE}}\right)^{3/2}.$$



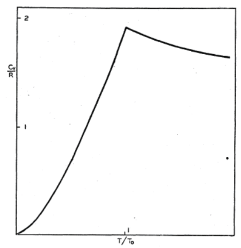
Specific heat of an ideal Bose-Einstein gas.

Note that the peak value is $1.926 k_B N$. This is obtained by setting $T = T_{BE}$, when $N_{ex} = N$ (all atoms are excited).

Above condensation temperature.

Above condensation temperature T_{BE}, we cannot use the same formula for the heat capacity. This is because it is derived assuming that the chemical potential $\mu \approx 0$, which is only true when condensation starts.

Above T_{BE}, μ changes. Albert Einstein has published a formula for this in 1924 (right half of curve):



Specific heat of an ideal Bose-Einstein gas.

At high temperatures, the heat capacity reaches the ideal gas value of $3Nk_B/2$.

The search for condensate.

After London's work, the search for BEC in liquid helium started. Calculations and experiments were carried to determine if BEC really existed in the superfluid.

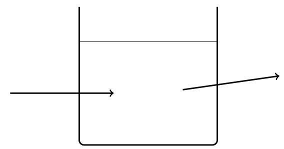
Measuring the amount of BEC is difficult. The method is to use neutron scattering to determine the velocities of the helium atoms.

In a BEC, the helium atoms are in the ground state - so their velocities are all zero. If the measurement can show that a significant fraction of the superfluid contains atoms with zero velocity, then we have found the condensate.

Neutron scattering.

A quick look at how neutrons can be used to determine the velocities of atoms would be helpful.

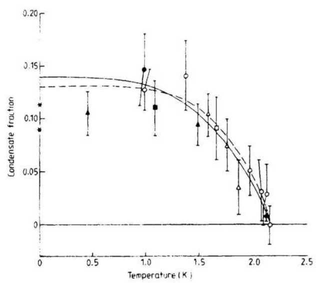
We cannot see the movement of the atoms in the superfluid, but we can measure the energy and direction of neutrons scattered from the superfluid.



Using momentum conservation, we can deduce the velocities of the helium atoms that the neutrons collided with.

Helium-4 condensate

The figure shows the theory and experimental results in the 1970s and 1980s. The condensate has indeed be found.



Left: Tilley and Tilley, Superfluidity and Superconductivity, (2003) p. 64.

However, even at 0 K, the condensate is only 10% of the superfluid. The reason is attributed to the interaction between atoms - it can only be 100% if the helium is an ideal gas at 0 K.

Helium-4 superfluid

Recall that London have proposed BEC as an explanation for superfluidity, because it behaves as a single wavefunction.

If BEC is only 10% of the superfluid, that means only 10% of the superfluid is a single wavefunction.

So why does 100% of liquid helium-4 behave as a superfluid at 0 K?

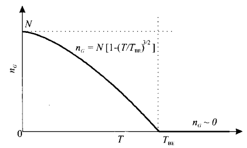
This problem is not fully understood until today.

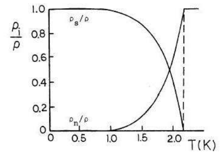
Even without a full understanding, it is still possible to make models that are useful for predicting the behaviour. This would in turn provide some understanding.

One month after London proposed the BEC theory for liquid helium in 1938, Laszlo Tisza proposed a two-fluid model for the superfluid.

The idea is that below the 2.18 K transition temperature, normal and superfluid coexist. As temperature falls, the normal fluid decreases. At 0 K, the superfluid increases to 100%.

The two-fluid model is closely related to the BEC idea. Recall the left figure, showing the condensate going to 100% as temperature falls from T_{BE} to 0 K.





Right: `http://www.yutopian.com/Yuan/TFM.html`

If we include the graph for the excited (not condensate) fraction, it would look like the figure on the right. This is just what the two-fluid model would suggest.

"According to this model, helium II behaves as if it were a mixture of two completely interpenetrating fluids with different properties, although in reality this is not the case.

To avoid any misunderstanding, it must be clearly stated at the outset that the two fluids cannot be physically separated; it is not permissible even to regard some atoms as belonging to the normal fluid and the remainder to the superfluid component, since all 4He atoms are identical."

In this model, the fluids are supposed to have the following properties:

	Normal fluid	*Superfluid*
Density	ρ_N	ρ_S
Velocity	$\mathbf{v}_N$	$\mathbf{v}_S$
Viscosity	η_N	$\eta_S = 0$
Entropy	S_N	$S_S = 0$

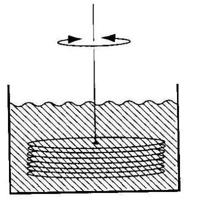
Guenault, Basic Superfluids (2003) p. 29

Note that we are not supposed to think that we can physically separate normal fluid from superfluid, since the helium-4 atoms are not distinguishable.

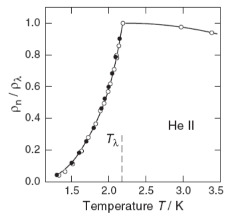
This makes it look rather unrealistic. However, following these assumptions, it has been possible both to explain many experiments.

Andronikashvili's experiment.

In 1948, Elepter Andronikashvili measured the fraction of normal fluid in liquid helium using a stack of rotating discs (left). The small spacings between discs means that any normal fluid would be dragged along and slow down the oscillation.



Andronikashvili's experiment.

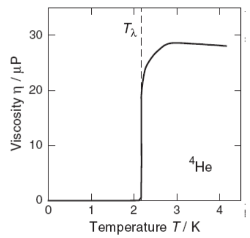


Right: Lifshitz and Andronikashvili, A Supplement to Helium (1959)

In the experiment, the oscillation period increases a lot below 2.2 K. By attributing this to a decrease in normal fluid in the spacing, the fraction of normal fluid can be deduced (right).

Viscosity measurement.

Viscosity can be measured by allowing liquid helium to flow through a very narrow capillary tube. From the velocity of flow, the viscosity can be calculated. The result is shown in this graph:



J.F. Allen, A.D. Misener, Proc. R. Soc. A172, 467 (1939)
K.R. Atkins, Philos. Mag. Supp. 1, 169 (1952)

Notice that viscosity drops to zero below 2.2 K.

Boiling helium.

When pressure is reduced below vapour pressure, superfluid helium boils (even if temperature is below 2.2 K).

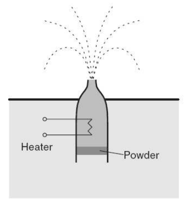
When a liquid like water boils, we expect to see bubbles forming the liquid moving.

When superfluid helium boils, we cannot see anything happening!

The helium atoms leave the surface of the liquid. Somehow, there is no bubble and no movement at all.

Fountain effect.

In the setup below (left), liquid helium in a flask and in a bath is separated by compressed powder which is porous.



Left: Enss and Hunklinger, Low-Temperature Physics (2005) p. 21. Right: J.F. Allen, photographed 1971, unpublished

The temperature is about 1 K. When the liquid in the flask is warmed very slightly with a heater, it shoots out of the flask like a fountain (right).

We have seen 2 viscosity experiments.

In the rotating discs experiment by Andronikashvili, viscosity decreases gradually as temperature falls below 2.2 K. This shows that the superfluid fraction increases gradually.

In the capillary tube experiment, the viscosity drops suddenly to zero. This suggests that the liquid suddenly becomes 100% superfluid.

The reason is that the capillary tube is too narrow. The normal fluid would experience too much viscosity trying to go through. So only the superfluid fraction passes through, and this experiences zero viscosity.

In the fountain experiment, when the liquid helium in the flask is warmed, the liquid shoots out like a fountain.

Using the two-fluid model, when it is warmed, the superfluid fraction changes to normal fluid. There is now a difference in concentration of superfluid between the flask and the helium bath below.

Since they are separated by compacted powder, superfluid can pass. Normal fluid cannot because of viscosity.

Since the concentration of superfluid in the flask decreased, superfluid will diffuse through the pores in the compacted powder from the bath. The volume of liquid in the fask then increases quickly and shoots out like a fountain.

When water boils, there is plenty of bubbles and movements. This happens because certain spots in the water gets a lot hotter than other parts. Boiling then starts a these hotspots, where bubbles are formed.

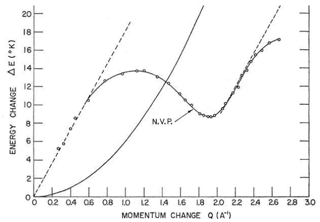
If bubbles are not formed in boiling helium, it must be because there are no hotspots. This can only be possible if the liquid has extremely good thermal conductivity. Any heat around potential hotspots must be conducted away immediately.

If any spot is hotter than the surrounding, it could convert the nearby superfluid into normal fluid. The concentration of superfluid decreases. Superfluid from further away flows in to the hotspot, and the normal fluid flows away.

This process is very efficient because the superfluid has no viscosity. The flow is limited only by the viscosity of the normal fraction.

As a result, heat conduction is extremely fast, and no hotspot can form. This is why no bubble is produced and the superfluid helium boils silently.

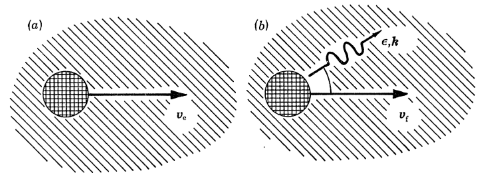
Landau's dispersion curve is used to deduce a critical velocity below which liquid helium is superfluid.



Henshaw and Woods, Physical Review, volume 121, p. 1266 (1961)

The above graph is measured using neutron scattering. Let us now see if the critical velocity calculated from this is indeed real.

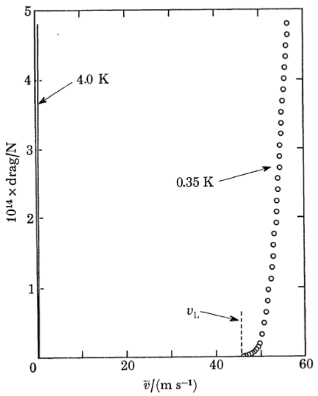
The idea is to accelerate a body through superfluid He-4, and see if it experiences a drag when the critical velocity is reached.



Allum, et al, Philosophical Transactions of the Royal Society A, vol. 284 (1977), p. 179

The idea is to inject a helium ion into the superfluid. This ion would attract surrounding atoms into a ball with about 100 atoms. It is then accelerated using an electric field.

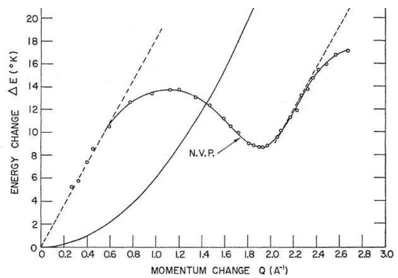
The drag measured is plotted against velocity of the ion.



Allum, et al, Philosophical Transactions of the Royal Society A, vol. 284 (1977), p. 179

The result shows that the critical velocity is about 45 m/s.

From the measured dispersion,



the following quantities are obtained:

$$\text{excitation: } \frac{\Delta}{k_B} = 8.65 \text{ K},$$

$$\text{momentum: } \frac{p_0}{\hbar} = 19.1 \text{ nm}^{-1}.$$

The Landau critical velocity is then given by

$$v_L = \frac{\Delta}{p_0} = 58 \text{ m s}^{-1}.$$

This is fairly close to the velocity of 45 m/s measured using the helium ion, and provides support for Landau's theory.

We now look at what the idea of a macroscopic wavefunction means for rotational motion in a superfluid.

Consider a simple case of uniform motion in one direction. The wavefunction may be represented by a plane wave:

$$\psi = e^{ikx}.$$

To see that the velocity is indeed constant, we use the equation in quantum mechanics for finding momentum:

$$-i\hbar\frac{d\psi}{dx} = p_x\psi.$$

Substituting the wavefunction and differentiating, we find

$$p_x = \hbar k.$$

This is the familiar relation for momentum in quantum mechanics. Dividing by mass of the helium atom m gives the velocity:

$$v = \frac{\hbar k}{m}.$$

In this simple wavefunction,

$$\psi = e^{ikx},$$

kx is the phase. In other wavefunctions, the phase can depend in a more complex way on the position x. Let the phase be $\phi(x)$.

The phase is useful to help explain the behaviour of the superfluid.

First, note that the simple wavefunction, as we move along x, the phase changes. A change of 2π corresponds to one wavelength.

Next, let us redo the calculation on the previous slide in terms in terms of the phase.

First, replace the phase in e^{ikx} by the more general function $\phi(x)$:

$$\psi = e^{i\phi(x)}.$$

Then, applying the equation in quantum mechanics for momentum:

$$-i\hbar\frac{d\psi}{dx} = p\psi.$$

we find

$$p = \hbar\frac{d\phi}{dx}.$$

Dividing by mass of the helium atom m gives the velocity:

$$v = \frac{\hbar}{m}\frac{d\phi}{dx}.$$

Next, consider what happens to the phase in a circular motion.

Suppose the superfluid is flowing round in a circle. Imagine following a path along this circle. When we return to the same point, the wavefuncton

$$\psi = e^{i\phi(x)}$$

must return to the same value. Then the phase ϕ must either change by 0, or by a multiple of 2π.

Let us now apply this idea to the velocity equation:

$$v = \frac{\hbar}{m}\frac{d\phi}{dx}.$$

When we integrate along the circular path, we get

$$\int \mathbf{v}.d\mathbf{l} = \frac{\hbar}{m}\Delta\phi$$

where $\Delta\phi$ is the change in phase, and $d\mathbf{l}$ is like a very short vector along the path.

In this equation,

$$\int \mathbf{v}.d\mathbf{l} = \frac{\hbar}{m}\Delta\phi.$$

the dot product is used on the left hand side. This is because we can think of the wavefunction like waves on the sea.

Only moving along direction $\mathbf{v}$ of the wave causes phase change. Moving along the phase front perpendicular to $\mathbf{v}$ gives no change in phase. The dot product picks out the component along $\mathbf{v}$.

We have seen that $\Delta\phi$ must be 0 or a multiple of 2π, so the equation becomes

$$\int \mathbf{v}.d\mathbf{l} = \frac{\hbar}{m}2n\pi = n\frac{h}{m},$$

where n is an integer.

In this equation

$$\int \mathbf{v}.d\mathbf{l} = n\frac{h}{m}$$

the left hand side is called the circulation. Since n is an integer, this means that the circulation is quantised.

Note the circulation is a measure of the amount of rotation. To see this, start with the simplest case when there is no rotation.

Then the velocity $\mathbf{v}$ is uniform, with the same magnitude and direction in the whole fluid. Integrating along a circle, we find that half the time, we move in the direction of flow, the other half we move opposite to the flow. Since there is a dot product in the circulation integral, that means the integral would be zero.

Next, consider uniform rotation of the superfluid. When we integrate along a circle, $\mathbf{v}$ and $d\mathbf{l}$ are always in the same direction. Then the integral would be nonzero. Also, if $\mathbf{v}$ is larger, then the integral is larger.

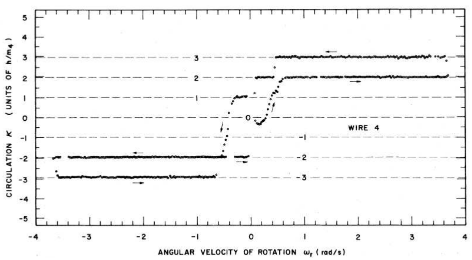
When water rotates in a bucket, is looks like this:

We call this a vortex. It is difficult to imagine a vortex like this can only rotate at certain speeds. Yet, this is exactly what London predicted with the wavefunction, if we have a bucket of superfluid helium.

Subsequently experiments have shown that this is indeed true.

Measuring vortices.

One way to measure circulation is to look at the vibration of a wire in a superfluid. When the circulation changes, the vibration changes. Using this method, the circulation is plotted against angular velocity:



Karn, et al, Physical Review B, vol. 21 (1980), p. 1797.

The distinct steps show that the circulation can only take discrete values, i.e. it is quantised.

What we have learnt so far.

1. Below 2.2 K, liquid helium-4 loses all viscosity. This new phase is called a superfluid.

2. London used Bose-Einstein condensation to explain this. He calculated a transition temperature of 3.13 K.

3. Theory and experimental search show that a condensate really does exist in superfluid helium-4, but only at 10%.

4. The two-fluid model can be used to explain observations like silent boiling, fountain effect, and zero viscosity.

5. Landau's prediction of a critical velocity is directly verified by experiments of moving ions.

6. Rotational motion in superfluid helium is quantised.

Exercises

Exercise 1

Bose-Einstein condensation is achieved with a gas of rubidium atoms, which have an atomic weight of 85.47. The condensation temperature is 1.7×10^{-7} K. What was the number density of the condensate?

[Atomic mass unit u is 1.6605×10^{-27} kg.]

Exercises

Recall the formula for the condensation temperature:

$$T_{BE} = \frac{h^2}{2\pi m k_B}\left(\frac{N}{2.612V}\right)^{3/2}$$

We can rearrange this to get the number density:

$$\frac{N}{V} = 2.612\left(\frac{2\pi m k_B T_{BE}}{h^2}\right)^{3/2}$$

The condensation temperature is given as

$$T_{BE} = 1.7 \times 10^{-7}\ \text{K}.$$

The mass is given by

$$m = 85.47u = 85.47 \times (1.6605 \times 10^{-27})\ \text{kg}.$$

Substituting these into the above equation for the number density, we get $2.74 \times 10^{19}\ \text{m}^{-3}$.

Exercise 2

Show that below the condensation temperature, T_{BE}, the heat capacity of a gas obeying Bose-Einstein statistics is given by

$$C_V = 1.93 N k_B \left(\frac{T}{T_{\text{BE}}}\right)^{3/2}.$$

[You are given that

$$\int_0^\infty \frac{x^{3/2}}{e^x - 1} dx = 1.78.$$

]

Recall that at very low temperatures, the chemical potential μ is approximately zero, and the number of particles above the ground state is given by:

$$N_{ex} = \int_0^\infty \frac{g(\varepsilon) d\varepsilon}{\exp(\varepsilon/k_B T) - 1}.$$

To obtain the energy, we insert ε into the integral:

$$U = \int_0^\infty \frac{g(\varepsilon)\varepsilon d\varepsilon}{\exp(\varepsilon/k_B T) - 1}.$$

Substituting the density of states

$$g(\varepsilon) = \frac{4m\pi V}{h^3}(2m\varepsilon)^{1/2}$$

gives

$$U = \frac{4m\pi V}{h^3} \int_0^\infty \frac{\sqrt{2m}\varepsilon^{3/2} d\varepsilon}{\exp(\varepsilon/k_B T) - 1}.$$

Define

$$x = \frac{\varepsilon}{k_B T}$$

and substitute into the integral for energy:

$$U = \frac{4m\pi V}{h^3}\sqrt{2m}(k_B T)^{5/2} \int_0^\infty \frac{x^{3/2} dx}{e^x - 1}.$$

We can now use the given result:

$$\int_0^\infty \frac{x^{3/2}}{e^x - 1} dx = 1.78.$$

This gives:

$$U = \frac{4m\pi V}{h^3}\sqrt{2m}(k_B T)^{5/2} \times 1.78.$$

The heat capacity can then be obtained by differentating this with respect to temperature. First, we express this in terms of the condensation temperature.

The condensation temperature is given by:

$$T_{BE} = \frac{h^2}{2\pi m k_B}\left(\frac{N}{2.612V}\right)^{3/2}$$

Make V the subject

$$V = \frac{1}{2.61}\left(\frac{h^2}{2\pi m k_B T_{BE}}\right)^{3/2}$$

and substitute into the above result for U:

$$U = \frac{2}{\sqrt{(\pi)}}\frac{1.78}{2.61} N k_B \frac{T^{5/2}}{T_{BE}^{5/2}}$$

Then differentiating with respect to T:

$$C_V = 1.93 N k_B \left(\frac{T}{T_{BE}}\right)^{3/2}$$

we get the heat capacity for the Bose-Einstein condensate.